Nonlinear Analysis: Real World Applications 63 (2022) 103422

Contents lists available at ScienceDirect

Nonlinear Analysis: Real World Applications

www.elsevier.com/locate/nonrwa

Linearizability of planar polynomial Hamiltonian systems

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A R T I C L E   I N F O

Article history:
Received 18 November 2020
Received in revised form 22 August 2021
Accepted 29 August 2021
Available online xxxx

Keywords:
Isochronicity
Linearizability
Planar polynomial differential systems
Hamiltonian systems
Integrability

A B S T R A C T

Isochronicity and linearizability of two-dimensional polynomial Hamiltonian systems are revisited and new results are presented. We give a new computational procedure to obtain the necessary and sufficient conditions for the linearization of a polynomial system. Using computer algebra systems we provide necessary and sufficient conditions for linearizability of Hamiltonian systems with homogeneous non-linearities of degrees 5, 6 and 7. We also present some sufficient conditions for systems with nonhomogeneous nonlinearities of degrees two, three and five.

1. Isochronicity of real planar polynomial Hamiltonian systems

A real two-dimensional system of differential equations is Hamiltonian if there exists a function H called a Hamiltonian, such that the system is written in the form

$$\dot{u} = -H_v(u,v), \quad \dot{v} = H_u(u,v). \tag{1.1}$$

Obviously, in this case we have that the operator $\mathcal{X} := \dot{u}\partial/\partial u + \dot{u}\partial/\partial v$ applied to H, that is $\mathcal{X}H$, is identically zero and consequently $H(u,v)$ is the first integral of system (1.1). In the case when the Hamiltonian function H is a polynomial, system (1.1) is a polynomial system and the degree of system (1.1) is the maximal degree of polynomials $H_u(u,v)$ and $H_v(u,v)$. Assume that system (1.1) has a nondegenerate center at the origin. Then there is an affine change of coordinates and a rescaling of time such that system (1.1) can be written in the form

$$\dot{u} = -v + U(u,v), \quad \dot{v} = u + V(u,v). \tag{1.2}$$

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https://doi.org/10.1016/j.nonrwa.2021.103422

Already Poincaré and Lyapunov characterized nondegenerate centers through the following result.

Theorem 1. *For any analytic system (1.2) having a center at the origin there exists an analytic function Ψ and an analytic change of coordinates of the form $\tilde{u} = u + o(|(u,v)|), \tilde{v} = v + o(|(u,v)|)$ that transforms the system to the form*

$$\dot{\tilde{u}} = -\tilde{v}(1 + \Psi(\tilde{u}^2 + \tilde{v}^2)), \quad \dot{\tilde{v}} = \tilde{u}(1 + \Psi(\tilde{u}^2 + \tilde{v}^2)). \tag{1.3}$$

We recall that a center of system (1.2) is isochronous, if all trajectories in some neighborhood of the origin have the same period. The next theorems give different methods to characterize the isochronous centers.

Theorem 2. *System (1.3) is isochronous with the period 2π if and only if*

$$T(r) = \int_0^{2\pi} \frac{d\theta}{1 + \Psi(r\cos\theta, r\sin\theta)} = 2\pi \ . \tag{1.4}$$

Theorem 3. *The origin of system (1.2) is isochronous if and only if there exists an analytic change of coordinates of the form $\tilde{u} = u + o(|(u,v)|), \tilde{v} = v + o(|(u,v)|)$ reducing the system to the linear system*

$$\dot{\tilde{u}} = -\tilde{v}, \quad \dot{\tilde{v}} = \tilde{u}, \tag{1.5}$$

that is, the origin of system (1.2) is an isochronous center if and only if system (1.2) is linearizable.

Theorem 4. *Let (S) and (S_T) be transversal plane differential systems of class C^2. Assume that the local flows defined by the solutions of (S) and (S_T) commute (in the sense of the Lie bracket). Then, any center of (S) is isochronous.*

Theorems 1–3 go back to Poincaré [1] and Lyapunov [2], the proofs can be also found e.g. in [3,4], Theorem 4 is proved in [5].

In the last decades many authors studied the isochronicity of centers, but a very few families have been completely characterized. In the case of systems with homogeneous nonlinearities, the quadratic, cubic and quintic isochronous centers were classified in [6–8], respectively. The case of systems with quartic homogeneous nonlinearities remains open and only partial results have been given, see [9,10]. For systems with nonhomogeneous nonlinearities the systems of degree three are still not solved, see for instance [11–13] for the classifications in some particular cases. These last works use the technique of the complexification of system (1.2), which is recalled in the next section.

In this paper we focus on polynomial Hamiltonian systems. Regarding the real polynomial Hamiltonian systems, from the results of Loud [6] and Peshkan [7] we know that there are no quadratic and cubic isochronous Hamiltonian centers with homogeneous nonlinearities. In the nineties of last century several authors [14–16] have proved that there are no real Hamiltonian systems with homogeneous nonlinearities which have isochronous centers. The real cubic polynomial Hamiltonian isochronous centers were classified in [17,18]. Following [18] we recall the notion of trivial centers.

Definition 1. Assume that system (1.2) has an isochronous center at the origin. The center is called *trivial*, if the Hamiltonian function $H(u,v)$ can be written in the form $H(u,v) = \frac{1}{2}(U(u,v)^2 + V(u,v)^2)$, where $U(u,v)$ and $V(u,v)$ are polynomials.

In fact in [18] it was shown that all isochronous centers of real cubic Hamiltonian systems are trivial. The result obtained was:

Theorem 5. *A real cubic Hamiltonian system of the form (1.2) has an isochronous center at the origin if and only if after a linear change of coordinates its Hamiltonian can be written as*

$$H(u,v) = (k_1 u)^2 + (k_2 v + P(u))^2, \tag{1.6}$$

where $k_1 \neq 0 \neq k_2$ and $P(u) = k_3 u + k_4 u^2$.

Open problem. [19] Are there any planar polynomial Hamiltonian systems of even degree with isochronous centers?

In [20] the following result has been proved.

Theorem 6. *The Hamiltonian differential system (1.2) has an isochronous center of period 2π at the origin if and only if its Hamiltonian function is of the form*

$$H(u,v) = \frac{1}{2}(U(u,v)^2 + V(u,v)^2), \tag{1.7}$$

the map

$$(u,v) \to (U(u,v), V(u,v)) \tag{1.8}$$

with $U(0,0) = V(0,0) = 0$, defined in some neighborhood of the origin is analytic and its Jacobian is constant and equal to one.

The authors of [19] argue that the previous result supports a negative answer to the open problem because in the case that $H(u,v)$ is a polynomial of odd degree it seems that there is some kind of obstruction for the existence of such a mapping.

In [21] it was proved that there are no planar polynomial Hamiltonian systems with nonlinearities of only even degrees and an isochronous center at the origin. Recently it has been proved in [22] that planar Hamiltonian differential systems of even degree with analytical mappings (1.8) defined on the whole plane do not have any isochronous center. In particular if the functions $U(u,v)$ and $V(u,v)$ are polynomials they are defined in the whole plane, consequently the analytic map $(u,v) \to (U(u,v), V(u,v))$ converges in all the plane.

We have presented above some main results on the isochronicity of real polynomial Hamiltonian systems. As we see only systems up to degree three are well investigated. One of obstacles in the classification of isochronous polynomial systems of higher degree is the extremely laborious computations of necessary conditions of isochronicity.

In several works, see for instance [8,10,13,23], it was shown that instead of studying isochronicity of real system (1.2) it is computationally more convenient to study the linearizability of the complex system associated to (1.2). The associated complex system is obtained from real system (1.2) with the following procedure.

We introduce a complex structure on the phase plane (u,v) by the substitution $x = u + iv$ and obtain from system (1.2) the complex differential equation

$$\dot{x} = R(x, \overline{x}).$$

Next we adjoin to this equation its complex conjugate and we have the system

$$\dot{x} = R(x, \overline{x}), \quad \dot{\overline{x}} = \bar{R}(x, \overline{x}).$$

Now we consider $\overline{x}$ as a new variable y and $\bar{R}$ as a new function obtaining a system of two complex differential equations. Then we can write the new system in the form

$$\begin{aligned}\dot{x} =& i(x - \sum_{p+q=1}^{n-1} a_{pq} x^{p+1} y^q),\\ \dot{y} =& -i(y - \sum_{p+q=1}^{n-1} b_{qp} x^q y^{p+1}),\end{aligned} \tag{1.9}$$

where $x, y \in \mathbb{C}$, $p \geq -1, q > 0$. In (1.9) a_{pq} and b_{qp} are independent parameters, however if it holds that

$$a_{pq} = \bar{b}_{qp} \tag{1.10}$$

then the complex line $y = \bar{x}$ is invariant for system (2.13). Viewing the line as a plane in $\mathbb{R}^4$ the flow on it is precisely the original flow of system (1.2) on $\mathbb{R}^2$ (see [4, §3.2]) for more details).

In the rest of the paper we study the linearizability of complex systems of the form (1.9) and obtain conditions for linearizability of certain families of complex polynomial Hamiltonian systems. Since by Theorem 3 isochronicity of a real system (1.2) is equivalent to its linearizability, imposing on the conditions of linearizability of complex systems condition (1.10) one can derive conditions of isochronicity of the corresponding embedded real systems.

2. Linearizability of complex planar systems

Consider a system of ordinary differential equations in the form

$$\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{X}(\mathbf{x}), \tag{2.1}$$

where $A = \operatorname{diag}\{\kappa_1, \dots, \kappa_n\}$, $\mathbf{x} = (x_1, \dots, x_n)$, $\mathbf{X}(\mathbf{x}) = (X_1(\mathbf{x}), \dots, X_n(\mathbf{x}))^T$, and $X_i(\mathbf{x})$ are complex convergent series which expansions start with at least quadratic terms.

Definition 2. It is said that system (2.1) is linearizable if there is an analytic transformation

$$\mathbf{x} = \mathbf{y} + \mathbf{h}(\mathbf{y}), \tag{2.2}$$

where the series in $\mathbf{h}(\mathbf{y}) = (h_1(\mathbf{y}), \dots, h_n(\mathbf{y}))^T$ do not contain constant and linear terms, defined on a neighborhood of the origin which brings (2.1) to the linear system

$$\dot{\mathbf{y}} = A\mathbf{y}.$$

Let $\kappa = (\kappa_1, \dots, \kappa_n)$, $\mathbb{N}_+ = \mathbb{N} \cup 0$. For $\alpha = (\alpha_1, \dots, \alpha_n)$ with $\alpha_i \in \mathbb{N}_+$ denote $(\kappa, \alpha) = \sum_{i=1}^n \alpha_i \kappa_i$ and $|\alpha| = \alpha_1 + \dots + \alpha_n$.

As it is well-known (see e.g. [1,24–27]) there is a substitution (2.2) which transforms system (2.1) to its Poincaré–Dulac normal form, that is, to a system of the form

$$\dot{\mathbf{y}} = A\mathbf{y} + \mathbf{Y}(\mathbf{y}), \tag{2.3}$$

where $\mathbf{Y}(\mathbf{y}) = (Y_1(y), \dots, Y_n(y))^T$ and Y_k $(k = 1, \dots, n)$, contain only terms of the form $Y_k^{(\alpha)} \mathbf{y}^\alpha$, where $(\kappa, \alpha) - \kappa_k = 0$. Terms $Y_k^{(\alpha)} \mathbf{y}^\alpha$ of $Y_k(\mathbf{y})$ and $h_k^{(\alpha)} \mathbf{y}^\alpha$ of $h_k(\mathbf{y})$ for which $(\kappa, \alpha) - \kappa_k = 0$ are called the resonant terms. Thus, if all coefficients of the resonant terms in $\mathbf{Y}(\mathbf{y})$ are equal to zero and normalizing substitution (2.2) is convergent on a neighborhood of the origin, then it is said that system (2.1) is linearizable.

The coefficients $Y_m^{(\alpha)}$ of the normal form and the coefficients $h_m^{(\alpha)}$ of the normalizing transformation are determined recursively with respect to the degree $s = |\alpha|$ of the terms of $\mathbf{h}(\mathbf{y})$ and $\mathbf{Y}(\mathbf{y})$ from the equation

$$[(\alpha,\kappa)-\kappa_m]h_m^{(\alpha)} = {g_m}^{(\alpha)} - Y_m^{(\alpha)}, \tag{2.4}$$

where on each step ${g_m}^{(\alpha)}$ (with $|\alpha| = s$) is a known expression. More precisely,

$${g_m}^{(\alpha)} = \{X_m(\mathbf{y}+\mathbf{h}(\mathbf{y}))\}^{(\alpha)} - \sum_{j=1}^{n} \sum_{\substack{2\le|\beta|\le|\alpha|-1 \\ \alpha-\beta+e_j\in\mathbb{N}_+^n}} \beta_j\, h_m^{(\beta)}\, Y_j^{(\alpha-\beta+e_j)}, \tag{2.5}$$

where $\{X_m(\mathbf{y}+\mathbf{h}(\mathbf{y}))\}^{(\alpha)}$ denotes the coefficient of $\mathbf{y}^\alpha$ obtained after expanding $X_m(\mathbf{y}+\mathbf{h}(\mathbf{y}))$ in powers of $\mathbf{y}$, and $e_j = (0,\dots,0,\overset{j}{1},0,\dots,0) \in \mathbb{N}_+^n$ (see e.g. [25, §2], [4, §2.3] for details). Note that for $|\alpha| = 2$ the sum over β is empty, so that $g_m^{(\alpha)} = \{X_m(\mathbf{y}+\mathbf{h}(\mathbf{y}))\}^{(\alpha)}$, which reduces to $g_m^{(\alpha)} = {X_m}^{(\alpha)}$, since $\mathbf{X}$ and $\mathbf{h}$ begin with quadratic terms. For $|\alpha| > 2$, $|\beta| < |\alpha|$ and $|\alpha-\beta+e_j| < |\alpha|$ ensure that $g_m^{(\alpha)}$ is uniquely determined by (2.5).

From (2.4) it is obvious, that the resonant coefficients of the normalizing transformation can be chosen arbitrarily. It means the normalizing transformation and, therefore, the normal form, are not uniquely defined. The normalizing transformation where all resonant coefficients are set to zero is called the distinguished normalizing transformation.

The transformation (2.2) does not necessarily converge, so, generally speaking $\mathbf{h}$ and $\mathbf{Y}$ are formal power series. It was proved by Pliss [28] that if for some normalizing transformation $\mathbf{Y}(\mathbf{y}) \equiv 0$ and all nonzero elements among $(\kappa,\alpha)-\kappa_k$, where $|\alpha| > 1$, satisfy the condition $|(\kappa,\alpha)-\kappa_k| \ge C|\alpha|^{-\nu}$, with some constants $C > 0, \nu > 0$, then there is also a convergent transformation to normal form. That is, the system is linearizable.

Later on, Bruno [26,27] proved that if system (2.1) can be transformed to the normal form

$$\dot{\mathbf{y}} = A\mathbf{y}(1+S(\mathbf{y})), \tag{2.6}$$

with $S(\mathbf{y})$ being a scalar function, then also there is an analytic transformation of (2.1) to normal form (2.6). Clearly, the condition of Bruno extends the condition of Pliss. Following Walcher [29] we say that normal form (2.3) of system (2.1) satisfies the Pliss–Bruno condition if it has the shape (2.6).

In this paper we are interested in the linearizability of two-dimensional systems, so we consider only the case when the matrix A has the form $A = \operatorname{diag}\{p,-q\}$, where p and q are positive integers, that is, the system

$$\begin{aligned} \dot{x}_1 &= px_1 + X_1(x_1,x_2), \\ \dot{x}_2 &= -qx_2 + X_2(x_1,x_2). \end{aligned} \tag{2.7}$$

Clearly, in such case the Pliss–Bruno condition is equivalent to the condition that the function

$$G(\mathbf{y}) = qY_1(\mathbf{y}) + pY_2(\mathbf{y}) \tag{2.8}$$

is identically zero, $G(\mathbf{y}) \equiv 0$.

We now prove the following result which we will use in the next section. This theorem can be considered as an extension of the Poincaré result for nondegenerate centers that affirms that if there exists a formal first integral then there exists a local analytic first integral around it. Our result affirms that if there exist a formal first integral and a formal linearizable change for any of the two equations then there exists an analytic linearizing change for the system.

Theorem 7. *Assume that system (2.7) admits a formal first integral of the form*

$$\Psi(x_1, x_2) = x_1^q x_2^p + h.o.t. \tag{2.9}$$

and there is a formal substitution which linearizes one of equations (2.7), let us say for determinacy that there is a formal substitution of the form

$$x_2 = z_2 + \theta(x_1, z_2), \tag{2.10}$$

where $\theta(x_1, y_2)$ is a series without constant and linear terms, which brings (2.7) to the form

$$\begin{aligned} \dot{x}_1 =& px_1 + X_1(x_1, z_2), \\ \dot{z}_2 =& -qz_2. \end{aligned} \tag{2.11}$$

Then system (2.7) is linearizable.

Proof. Let

$$x_1 = y_1 + h_1(y_1, y_2), \quad z_2 = y_2 + h_2(y_1, y_2)$$

be a normalizing transformation of system (2.11). From (2.4) and (2.5) we see that it is possible to take $h_2(y_1, y_2) \equiv 0$. Then the normal form of (2.11) is

$$\begin{aligned} \dot{y}_1 =& py_1 + Y_1(y_1, y_2), \\ \dot{y}_2 =& -qy_2. \end{aligned} \tag{2.12}$$

By (2.9) and (2.10) system (2.12) has a formal first integral of the form $\Psi_1(x_1, z_2) = x_1^q z_2^p + h.o.t.$ Therefore, according to [4, Theorem 3.2.5] for the normal form (2.12) the function $G(\mathbf{y})$ defined by (2.8) is identically zero. Since in (2.12) $Y_2(y_1, y_2) \equiv 0$ it yields that in (2.12) $Y_1(y_1, y_2) \equiv 0$. That is, system (2.7) can be transformed to a linear system by means of a formal normalizing transformation. By the Pliss theorem [28] there is also an analytic transformation of (2.7) to the linear system, that is, system (2.7) is linearizable. □

We now study the linearizability of two-dimensional system (1.9). After rescaling of time $idt = d\tau$ and writing again t instead of τ, we obtain the system

$$\begin{aligned} \dot{x} =& x - \sum_{p+q=1}^{n-1} a_{pq} x^{p+1} y^q = P(x, y), \\ \dot{y} =& -y + \sum_{p+q=1}^{n-1} b_{qp} x^q y^{p+1} = Q(x, y), \end{aligned} \tag{2.13}$$

where $x, y \in \mathbb{C}$, $p \geq -1$, $q > 0$. Clearly, if a system in family (2.13) is linearizable, then the correspondent system (1.9) is linearizable as well, and vice versa.

The linearizability problem for system (2.13) is to examine if we can transform it into the linear system

$$\begin{aligned} \dot{z} =& z, \\ \dot{w} =& -w \end{aligned} \tag{2.14}$$

by a near-identity analytic transformation defined in a neighborhood of the origin. By Theorem 4.3.2 of [4] instead of a normalizing transformation we can, equivalently, look for an analytic change of variables of the form

$$\begin{aligned} z =& x + \sum_{m+j=2}^{\infty} u^{(1)}_{m-1,j}(a, b) x^m y^j, \\ w =& y + \sum_{m+j=2}^{\infty} u^{(2)}_{m,j-1}(a, b) x^m y^j, \end{aligned} \tag{2.15}$$

where $u_{ij}^{(k)}(a,b)$ are polynomial functions in $\mathbb{Q}[a,b]$, such that in the coordinates z, w the systems have the form (2.14).

To get the necessary conditions for existence of such transformation for systems in family (2.13) we first differentiate Eqs. (2.15) and equate the right-hand side to the right-hand side of (2.14). Then we equate the coefficients of similar terms on both sides of the obtained equations. This yields the recurrence formulas

$$
\begin{aligned}
(q_1-q_2)u^{(1)}_{q_1,q_2} &= \sum_{s_1+s_2=0}^{q_1+q_2-1}\big((s_1+1)u^{(1)}_{s_1,s_2}a_{q_1-s_1,q_2-s_2} - s_2u^{(1)}_{s_1,s_2}b_{q_1-s_1,q_2-s_2}\big),\\
(q_1-q_2)u^{(2)}_{q_1,q_2} &= \sum_{s_1+s_2=0}^{q_1+q_2-1}\big(s_1u^{(2)}_{s_1,s_2}a_{q_1-s_1,q_2-s_2} - (s_2+1)u^{(2)}_{s_1,s_2}b_{q_1-s_1,q_2-s_2}\big),
\end{aligned}
$$

where $s_1,s_2\ge -1$, $q_1,q_2\ge -1$, $q_1+q_2\ge 0$, $u^{(1)}_{1,-1}=u^{(1)}_{-1,1}=u^{(2)}_{1,-1}=u^{(2)}_{-1,1}=0$, $u^{(1)}_{0,0}=u^{(2)}_{0,0}=1$, $a_{c,d}=b_{c,d}=0$ for $c+d<1$.

For $q_1\neq q_2$ we can get the coefficients $u^{(1)}_{q_1,q_2}$ and $u^{(2)}_{q_1,q_2}$ of the transformation directly from the formulas above. But when $q_1=q_2$ (calculating coefficients of terms $x^{q_1+1}y^{q_2}$ from the first equation and $x^{q_1}y^{q_2+1}$ from the second one) the coefficients can be chosen arbitrarily. The most convenient choice for the further computations is to set them to zero. We denote the right-hand side of the first equation by i_{kk} and the right-hand side of the second equation by j_{kk} for $q_1=q_2=k$. The quantities i_{kk}, j_{kk} are called *k-th linearizability quantities* (see e.g. [4] for more details about calculations of the linearizability quantities).

A necessary condition for linearizability is then $i_{kk}=j_{kk}=0$ for all $k\in\mathbb{N}$. Linearizability quantities form the ideal

$$
\mathcal{L} := \langle i_{11}, j_{11}, i_{22}, j_{22}, \dots\rangle \subset \mathbb{C}[a,b], \tag{2.16}
$$

where a and b are the vectors of parameters of the first equation and second equations of (2.13), respectively. Let $\mathcal{L}_k$ be the ideal generated with the first k pairs of linearizability quantities. With the irreducible decomposition of the variety of the ideal $\mathcal{L}_k$, $\mathbf{V}(\mathcal{L}_k)$, for a sufficiently large k we can compute the components of the variety, which may define linearizable families of (2.13), that is, the necessary conditions for linearizability of the system.

Linearizability quantities are polynomials in variables a_{pq} and b_{qp} and their size increases exponentially with growing k. Consequently, the computation of the irreducible decomposition can be very difficult or impossible even for the most powerful computer algebra programs.

Remark 1. By Theorem 4.3.2 of [4] the variety $\mathbf{V}(\mathcal{L})$ is the same as the variety defined by the coefficients $Y_1^{(j+1,j)}, Y_2^{(j,j+1)}$ of a normal form of system (2.13), so, as it is mentioned above, to study the linearizability we can either look for a normalizing transformation or for its inverse (2.15).

Another method to find conditions of linearizability is to look for a polynomial linearization of one of equations of the system, for instance to look for a *linearization of the first equation* of (2.13) in the form

$$
X = x + \varphi(x,y), \tag{2.17}
$$

where φ is a polynomial without constant and linear terms,

$$
\varphi(x,y) = \sum_{i+j=2}^{k} \phi_{ij}x^iy^j. \tag{2.18}
$$

Let $\mathcal{X} = P\partial/\partial x + Q\partial/\partial y$ be a derivative with respect to vector field (2.13). Then substitution (2.17) linearizes the first equation of (2.13), if

$$
\mathcal{X}X = X. \tag{2.19}
$$

Then $X = 0$ is an invariant algebraic curve for (2.13) and since $H(x, y)$ is the Hamiltonian function for (2.13), $g = -H/X$ is an analytic function of the form $g(x, y) = y + \cdots$ and the second equation of system (2.13) is linearizable by the substitution $Y = g(x, y)$. Analyticity of the transformation follows from the analytic Nullstellensatz for principal ideals (Theorem 18 of [30, p. 90] — the reasoning is similar as in the proof of Proposition 1 in [31]).

Similar to what has been done in [22] to get conditions for linearizability of a family of system (2.13) we can equate the coefficients of the same monomials on both sides of (2.19) obtaining a polynomial system in the variables ϕ_{ij} and the parameters a_{ij} and b_{ij} of the system. Then using the routine `eliminate` of the compute algebra system Singular [32] we eliminate from the obtained polynomial system ϕ_{ij} and compute the primary decomposition of the obtained ideal deriving some necessary conditions for the linearizability.

We emphasize that a priori with this approach we may be not able to find all conditions for linearizability — *we look only for systems in which at least one equation is linearizable by a polynomial substitution.*

However if we take φ as a power series, then theoretically we should find all conditions for linearizability and not only particular cases as in the situation when we choose φ as a polynomial. Indeed, if we look for the linearization of the first equation in the form

$$X = x + \sum_{i+j\geq 2} \phi_{ij} x^i y^j,$$

then equating the coefficients of the monomials up to degree k of the series

$$\mathcal{X}X - X$$

to zero we obtain a system, let us say system (S_k), of polynomial equations in the variables ϕ_{ij} and a_{pq}, b_{qp}. We denote by I_k the ideal generated by the polynomials defining system (S_k) in the ring

$$\mathbb{C}[a, b, \phi_k],$$

where a and b are as in (2.16) and ϕ_k is the vector of parameters ϕ_{ij} such that $i + j \leq k$. If for some fixed values a^*, b^* of parameters a, b system (2.13) is linearizable then for any $k \geq 2$ the system (S_k) has a solution. Therefore the coefficients a^*, b^* of the system belong to the variety of the elimination ideal

$$J_k = I_k \cap \mathbb{C}[a, b].$$

Let

$$J = \cup_{k\geq 2} J_k \subset \mathbb{C}[a, b].$$

Since the ring $\mathbb{C}[a, b]$ is Noetherian by the Hilbert basis theorem there is a natural number k_0 such that

$$J = J_{k_0}.$$

Then the variety V of J_{k_0} gives the necessary conditions for linearizability. Thus, we have established the following statement.

Theorem 8. *There exists a polynomial (2.18) of certain degree k_0 such that the corresponding elimination ideal*

$$J_{k_0} = I_{k_0} \cap \mathbb{C}[a, b]$$

gives the necessary linearizability conditions for polynomial system (2.13).

The problem with the application of the above theorem is that we do not know the value of k_0, but still the theorem gives a computational procedure to obtain the conditions for linearization of a polynomial system (2.13).

In practice, to find conditions for linearizability one can:

(1) Compute the ideals $J_1, J_2, \dots$ until for some q $\sqrt{I_q} = \sqrt{I_{q-1}}$ (where the square root stands for the radical of the ideal).

(2) Compute the irreducible decomposition of the variety of I_{q-1}.

(3) For each component of the irreducible decomposition prove existence of a linearization (the sufficiency) of the obtained conditions.

(4) If some ideal $\mathcal{L}_m := \langle i_{11}, j_{11}, i_{22}, j_{22}, \dots, i_{mm}, j_{mm} \rangle$ is also computed then one can also check if

$$\sqrt{L_m} = \sqrt{I_{q-1}}.$$

If this is the case then the obtained conditions are the necessary and sufficient conditions for linearizability.

The linearizability of complex differential systems is studied in several works, see for instance [8,10,13]. The classification of the linearizable systems with quadratic, cubic and quintic homogeneous nonlinearities is known, see [33] and [8]. As in the case of isochronicity of real systems, the classification of linearizable complex systems with quartic homogeneous nonlinearities is still open, see [10]. The linearizability of complex systems with nonhomogeneous nonlinearities of degree three is still not solved, see for instance [13] where a particular case is classified.

In this paper we focus on the complex Hamiltonian systems studying some families of complex polynomial Hamiltonian differential systems. We recall that the classification of the complex linearizable Hamiltonian systems with cubic nonlinearities is also unknown and only some subfamilies were found using the second method described above with a φ of fourth degree, see [22]. Moreover it was shown there that this method allows to detect all real cubic Hamiltonian polynomial systems. Using the method five subfamilies of linearizable complex cubic Hamiltonian systems were found, however it was unknown if they provide the complete list of linearizable systems in this family. We will see below that the list is not complete.

Remark 2. It is relevant to specify that the isochronicity problem of the real Hamiltonian systems is not equivalent to the linearizability problem of the complex Hamiltonian systems of the same degree. For instance, the equivalent above open problem: are there linearizable complex Hamiltonian systems of even degree?, has a positive answer in the complex Hamiltonian systems. In fact there exist linearizable complex Hamiltonian systems with quadratic homogeneous nonlinearities, see [22] and the beginning of Section 3.

In this work we study the linearizability problem for several families of complex polynomial Hamiltonian systems. For some of the studied families we were not able to complete computations using the first approach, however we have obtained some results following the second approach using a φ of a certain degree.

3. Linearizability of systems with homogeneous nonlinearities

We recall that the collection of Hamiltonian systems in a family of the form (2.13) are precisely those systems whose coefficients satisfy the following condition:

$$(p+1)a_{pq} = (q+1)b_{pq}. \tag{3.1}$$

As it is mentioned in Section 2 all linearizable systems in the form of the linear center perturbed by homogeneous polynomials of degrees two, three and five were classified in the previous works. Clearly, it includes also the cases of Hamiltonian systems. From the results of [33] it follows that the Hamiltonian quadratic system, that is, the system

$$\begin{aligned}\dot{x} =& x - a_{10}x^2 - 2b_{01}xy - a_{-12}y^2,\\ \dot{y} =& -y + b_{2,-1}x^2 + 2a_{10}xy + b_{01}y^2,\end{aligned} \tag{3.2}$$

is linearizable if and only if

$$b_{2,-1} = a_{10} = 0 \quad \text{or} \quad b_{01} = a_{-12} = 0,$$

and the Hamiltonian system with non-linearities of degree three, that is, the system

$$\begin{aligned}\dot{x} =& x - a_{20}x^3 - a_{11}x^2y - 3b_{02}xy^2 - a_{-13}y^3,\\ \dot{y} =& -y + b_{3,-1}x^3 + 3a_{20}x^2y + a_{11}xy^2 + b_{02}y^3,\end{aligned} \tag{3.3}$$

is linearizable if and only if one of the following conditions holds:

$$a_{-13} = b_{02} = a_{11} = 0 \quad \text{or} \quad b_{3,-1} = a_{20} = a_{11} = 0.$$

We note that recently some new results on the Hamiltonian systems with quadratic and homogeneous cubic nonlinearities have been obtained in [34].

The results of [8] applied to the case Hamiltonian systems yield that the Hamiltonian system with nonlinearities of degree five, that is, the system

$$\begin{aligned}\dot{x} =& x - a_{40}x^5 - a_{31}x^4y - a_{22}x^3y^2 - 2b_{13}x^2y^3 - 5b_{04}xy^4 - a_{-15}y^5,\\ \dot{y} =& -y + b_{5,-1}x^5 + 5a_{40}x^4y + 2a_{31}x^3y^2 + a_{22}x^2y^3 + b_{13}xy^4 + b_{04}y^5,\end{aligned} \tag{3.4}$$

is linearizable if and only if

$$a_{22} = a_{31} = a_{40} = b_{5,-1} = 0 \quad \text{or} \quad a_{22} = a_{-15} = b_{04} = b_{13} = 0.$$

The next theorems give the results of our studies for the case of planar complex Hamiltonian systems (that is, systems in the form (2.13) with condition (3.1)) in the case of homogeneous perturbations of degrees four, six and seven. We recall that two systems (2.13) are conjugate if one is obtained from the other after the change $x \leftrightarrow y, a_{ij} \leftrightarrow b_{ji}, t \to -t$. Clearly, if one of conjugate systems is linearizable, then the other is linearizable as well.

Theorem 9. *The Hamiltonian system (2.13) with homogeneous non-linearities of degree four is linearizable at the origin if and only if one of the following conditions holds:*

(1) $b_{4,-1} = a_{21} = a_{30} = 0$,
(2) $b_{03} = a_{-14} = a_{12} = 0$.

Proof. For this system we have computed the first seven non-zero pairs of the linearizability quantities. The first pair is

$$\begin{aligned}i_{11} =& 5a_{12}a_{21} + 5a_{03}a_{30} - 5a_{21}b_{12} + 10a_{12}b_{21} + 5a_{03}b_{30} + 4a_{14}b_{41},\\ j_{11} =& 5a_{21}b_{12} - 10a_{12}b_{21} - 5b_{12}b_{21} - 5a_{03}b_{30} - 5b_{03}b_{30} - 4a_{14}b_{41}.\end{aligned}$$

We do not present the other quantities here since their expressions are too long (the second pair has already 68 terms each and the number of terms grows exponentially).

Then, with the routine `minAssGTZ` [35] of the computer algebra system Singular [32], we found that the variety of the obtained ideal consists of two components given in the statement of the theorem.

In case (1) the corresponding system is

$$\begin{aligned}\dot{x} =& x - a_{12}x^2y^2 - 4b_{03}xy^3 - a_{-14}y^4,\\ \dot{y} =& -y + \frac{2}{3}a_{12}xy^3 + b_{03}y^4.\end{aligned} \tag{3.5}$$

After the substitution

$$u = xy^2, \quad v = y^3 \tag{3.6}$$

we obtain system

$$\dot{u} = -u + a_{12}/3u^2 - 2b_{03}uv - a_{-1,4}v^2, \qquad \dot{v} = -3v + 2a_{12}uv + 3b_{03}v^2. \tag{3.7}$$

By the Poincaré–Dulac normal form theory (see e.g. [1,24,25]) an analytic system of the form

$$\dot{u} = -u + \sum_{j+k=2} U_{jk}u^jv^k, \qquad \dot{v} = -nv + \sum_{j+k=2}^{\infty} V_{jk}u^jv^k,$$

by a convergent transformation

$$\xi = u + \sum_{j+k=2}^{\infty} \alpha_{jk}u^jv^k, \quad \eta = v + \sum_{j+k=2}^{\infty} \beta_{jk}u^jv^k, \tag{3.8}$$

can be brought to the normal form

$$\dot{\xi} = -\xi, \quad \dot{\eta} = -n\eta + a\xi^n. \tag{3.9}$$

Computing the normal form we find that for system (3.7) $a = 0$, that is, the system is linearizable by the substitution (3.8). Then, the substitution

$$X = \xi\eta^{-2/3}, \quad Y = \eta^{1/3}, \tag{3.10}$$

transforms (3.9) into $\dot{X} = X$, $\dot{Y} = -Y$. Substituting into (3.10) the expressions (3.6) and (3.8)) we see that the functions on the right hand sides of (3.10) are analytic functions of x and y. Thus, there is an analytic change of variables of the form

$$X = x + h.o.t., \quad Y = y(1 + h.o.t.),$$

which transforms the original system (3.5) into the linear system $\dot{X} = X, \dot{Y} = -Y$. The system of the second case is conjugate to (3.5). □

Theorem 10. *The Hamiltonian system with homogeneous non-linearities of degree six is linearizable at the origin if one of the following conditions holds:*

(1) $b_{6,-1} = a_{32} = a_{41} = a_{50} = 0$,
(2) $a_{-16} = a_{23} = a_{14} = b_{05} = 0$.

Proof. For the system under consideration we first calculated the first seven non-zero linearizability quantities. After the decomposition of the variety of the ideal, generated by these quantities, we got the two components given in the statement of the theorem.

The system corresponding to the first component is written as

$$\begin{aligned}\dot{x} =& x - a_{23}x^3y^3 - a_{14}x^2y^4 - 6b_{05}xy^5 - a_{-16}y^6,\\ \dot{y} =& -y + \frac{3}{4}a_{23}x^2y^4 + \frac{2}{5}a_{14}xy^5 + b_{05}y^6.\end{aligned} \tag{3.11}$$

For system (3.11) we have not found an explicit linearizing substitution but we prove its existence. We look for the linearization of the second equation of the system in the form of the series

$$z_2 = y + \sum_{k=2}^{\infty} f_k(x) y^k, \tag{3.12}$$

where $f_k(x)$ are some polynomials. Substituting (3.12) into the equation

$$\dot{z}_2 = -z_2$$

and equaling the coefficients of the same powers of y we see that the polynomials $f_k(x)$ satisfy the differential equations

$$\begin{aligned}(k-5)b_{05}f_{k-5}(x) + \frac{2}{5}(k-4)a_{14}xf_{k-4}(x) + \frac{3}{4}(k-3)a_{23}x^2 f_{k-3}(x) - (k-1)f_k(x) + \\ -\ 6b_{05}xf'_{k-5}(x) - a_{14}x^2 f'_{k-4}(x) - a_{23}x^3 f'_{k-3}(x) + xf'_k(x) - a_{-16}f'_{k-6}(x) = 0,\end{aligned} \tag{3.13}$$

where we initialize by setting $f_j \equiv 0$ for $j \leq 0$ and $f_1 = 1$.

Differential equation (3.13) for $k = 2$ is

$$-f_2 + xf'_2 = 0$$

which gives us

$$f_2 = C_2 x.$$

Then, for $k = 3$ we have

$$-2f_3 + xf'_3 = 0$$

and it yields

$$f_3 = C_3 x^2.$$

Moreover, direct calculations for $k = 4, \dots, 9$ show that

$$\begin{aligned}
f_4 =& \frac{3}{4}a_{23}x^2 + C_4x^3, f_5 = \frac{2}{15}a_{14}x + \frac{1}{2}a_{23}C_2x^3 + C_5x^4,\\
f_6 =& \frac{1}{5}b_{05} - \frac{1}{15}a_{14}C_2x^2 + \frac{1}{4}a_{23}C_3x^4 + C_6x^5, f_7 = -\frac{4}{5}b_{05}C_2x - \frac{4}{15}a_{14}C_3x^3 + \frac{3}{8}a_{23}^2x^4 + C_7x^6,\\
f_8 =& -\frac{1}{7}a_{-16}C_2 - \frac{9}{5}b_{05}C_3x^2 + \frac{1}{60}a_{14}a_{23}x^3 - \frac{7}{15}a_{14}C_4x^4 + \frac{3}{16}a_{23}^2C_2x^5 - \frac{1}{4}a_{23}C_5x^6 + C_8x^7,\\
f_9 =& -\frac{2}{7}a_{-16}C_3x + \frac{1}{45}a_{14}^2x^2 - \frac{17}{20}a_{23}b_{05}x^2 - \frac{14}{5}b_{05}C_4x^3 - \frac{1}{6}a_{14}a_{23}C_2x^4 - \frac{2}{3}a_{14}C_5x^5 + \\
& \frac{1}{16}a_{23}^2C_3x^6 - \frac{1}{2}a_{23}C_6x^7 + C_9x^8
\end{aligned}$$

and setting the integration constants $C_i, i = 2, \dots, 9$, we notice that functions f_i have the following shapes:

$$f_{3k+1} = p_{2k}, f_{3k+2} = p_{2k-1}, f_{3k+3} = p_{2k-2}, \tag{3.14}$$

where $k = 1, 2$ and p_n is a certain polynomial of degree n. Now we will prove by induction that (3.14) holds for all $k \in \mathbb{N}$. The induction basis for $k = 1$ is covered by

$$f_4 = \frac{3}{4}a_{23}x^2 = p_2, f_5 = \frac{2}{15}a_{14}x = p_1, f_6 = \frac{1}{5}b_{05} = p_0.$$

For the induction step we assume that (3.14) holds for all $n \leq k, n, k \in \mathbb{N}$ and we prove it holds also for $n = k + 1$. First, we observe functions of the form f_{3k+1}. We will prove that

$$f_{3(k+1)+1} = p_{2(k+1)}.$$

Solution for differential equation (3.13) is in this case

$$\begin{aligned} f_{3(k+1)+1} =& f_{3k+4} = x^{3(k+1)}C_{3k+4} + x^{3(k+1)}\int_1^x t^{-3(k+1)-1}((-3k+1)b_{05}f_{3k-1} - \frac{6}{5}ka_{14}f_{3k}t+ \\ &- \frac{3}{4}(3k+1)a_{23}f_{3k+1}t^2 + a_{14}f'_{3k}t^2 + 6b_{05}f'_{3k-1}t + a_{-16}f'_{3k-2} + a_{23}f'_{3k+1}t^3)dt. \end{aligned} \tag{3.15}$$

We set constant C_{3k+4} to zero and exponents in the expression in the brackets under the integral on the right side of Eq. (3.15) regarding the induction hypothesis are of degree $2k+2$ or lower, so we can write

$$f_{3(k+1)+1} = x^{3k+3}\int_1^x t^{-3k-4}p_{2k+2}(t)dt = x^{3k+3}\int_1^x p_{-k-2}(t)dt = x^{3k+3}p_{-k-1}(x) = p_{2k+2}(x),$$

(polynomials p_n in the above formulas can be different — we indicate only their degrees) which validates our assumption. In a similar manner we obtain the following results for the other two cases from (3.14):

$$f_{3(k+1)+2} = x^{3k+4}\int_1^x t^{-3k-5}p_{2k+1}(t)dt = p_{2(k+1)-1}(x),$$

$$f_{3(k+1)+3} = x^{3k+5}\int_1^x t^{-3k-6}p_{2k}(t)dt = p_{2(k+1)-2}(x).$$

Thus, the induction hypothesis holds and, therefore, (3.12) is a formal linearization of the second equation of (3.11).

Let $y = z_2 + \theta(x, z_2)$ be the inverse of (3.12). Then this substitution linearizes the second equation of (3.12). Since system (3.12) is Hamiltonian it admits a first integral of the form $H(x,y) = xy + h.o.t.$ Then by Theorem 7 system (3.12) is linearizable.

The second condition gives the system which is conjugate to system (3.11). □

Theorem 11. *The Hamiltonian system with homogeneous non-linearities of degree seven is linearizable at the origin if and only if one of the following conditions holds:*

(1) $a_{33} = a_{60} = a_{42} = a_{51} = b_{7,-1} = 0,$
(2) $a_{33} = b_{06} = a_{24} = b_{15} = a_{-17} = 0.$

Proof. We have calculated the first 12 pairs of linearizability quantities for the Hamiltonian system with homogeneous non-linearities of degree seven. Similarly as in the previous theorems, with help of Singular, we found that the necessary conditions for the linearizability are the two conditions in the statement of the theorem.

The first component yields the system

$$\begin{aligned} \dot{x} =& x - \frac{5}{3}b_{24}x^3y^4 - 3b_{15}x^2y^5 - 7b_{06}xy^6 - a_{-17}y^7, \\ \dot{y} =& -y + b_{24}x^2y^5 + b_{15}xy^6 + b_{06}y^7. \end{aligned} \tag{3.16}$$

Similarly as in the proof of Theorem 9 we apply substitution (3.6) and obtain system

$$\dot{u} = -u + \frac{b_{24}}{3}u^3 - b_{15}u^2v - 5b_{06}uv^2 - a_{-17}v^3, \qquad \dot{v} = -3v + 3b_{24}u^2v + 3b_{15}uv^2 + 3b_{06}v^3. \tag{3.17}$$

Again, the normal form of this system is (3.9) with $n = 3$ and the calculations show that $a = 0$, so the system is linearizable and substitution (3.10) yields $\dot{X} = X, \dot{Y} = -Y$.

The second condition gives the system which is conjugate to system (3.16), thus, we can find the linearization for it in a similar manner. □

4. Linearizability of systems with cubic nonhomogeneous nonlinearities

In [22] the classification of the complex linearizable Hamiltonian systems with cubic nonlinearities admitting a polynomial linearization of one of equations of the system is studied. Using a polynomial change of the form (2.18) with φ of fourth degree some linearizable subfamilies are given. The problem is to know if this classification is complete. Here we present the result when we use a polynomial φ of degree 6.

Theorem 12. *Consider the cubic Hamiltonian system*

$$\begin{aligned}\dot{x} &= x - a_{10}x^2 - a_{20}x^3 - a_{01}xy - a_{11}x^2y - a_{-12}y^2 - a_{02}xy^2 - a_{-13}y^3,\\ \dot{y} &= -y + b_{2,-1}x^2 + b_{3,-1}x^3 + b_{10}xy + b_{20}x^2y + b_{01}y^2 + b_{11}xy^2 + b_{02}y^3,\end{aligned} \tag{4.1}$$

where $b_{10} = 2a_{10}$, $a_{01} = 2b_{01}$, $a_{02} = 3b_{02}$, $b_{20} = 3a_{20}$, $b_{11} = a_{11}$. System (4.1) is linearizable at the origin if one of the following conditions holds:

(1) $b_{3,-1} = b_{2,-1} = a_{20} = a_{11} = a_{10} = 0$,
(2) $b_{02} = b_{01} = a_{-13} = a_{-12} = a_{11} = 0$,
(3) $27b_{01}^3 + a_{-12}^2 b_{2,-1} = 9a_{10}b_{01} - a_{-12}b_{2,-1} = a_{10}a_{-12} + 3b_{01}^2 = 3a_{10}^2 + b_{01}b_{2,-1} = -4/3a_{10}b_{2,-1} + b_{3,-1} = -4b_{01}^2 + b_{02} = 4/3b_{01}b_{2,-1} + a_{20} = -4/3a_{-12}b_{01} + a_{-13} = 4/3a_{-12}b_{2,-1} + a_{11} = 0$.

The cases (1) and (3) of Theorem 4 in [22] are included in (1) of Theorem 12. The cases (2) and (4) of Theorem 4 in [22] are included in (2) and the case (5) are of Theorem 4 in [22] corresponds to case (3). All the cases of Theorem 12 are time-reversible and Darboux linearizable. However in cases (1) and (2) the linearizing changes are not polynomial as can be seen in the proof of the theorem. Therefore, although we impose the existence of a polynomial change the method can find cases whose linearizing changes are not polynomial. We computed irreducible decomposition of the variety of the ideal $\mathcal{L}_7 = \langle i_{11}, j_{11}, i_{22}, j_{22}, \ldots, i_{77}, j_{77}\rangle$ when $a_{11} = 0$ and have obtained the first two components stated in Theorem 12. Our computational facilities have not been able to compute the irreducible decomposition for $a_{11} \neq 0$.

We have also computed the classification using the approach described in the previous section with φ defined by (2.18) of degree 7 and we have obtained the same cases that appear in Theorem 12. That is, using the procedure explained after Theorem 8, we have that the first three steps of it are satisfied. However, the last step, that is the verification that $\sqrt{\mathcal{L}_7} = \sqrt{I_7}$ has not been possible to complete. For this reason we cannot assure that the list of the linearizability conditions given in Theorem 12 is complete.

Proof. In case (1) the system takes the form

$$\begin{aligned}\dot{x} &= x - 2b_{01}xy - a_{-12}y^2 - 3b_{02}xy^2 - a_{-13}y^3,\\ \dot{y} &= y(-1 + b_{01}y + b_{02}y^2)\end{aligned}$$

It has three invariant lines which are $\ell_1 = y$, $\ell_2 = 1 + (-b_{01} - \sqrt{b_{01}^2 + 4b_{02}})y/2$ and $\ell_3 = 1 + (-b_{01} + \sqrt{b_{01}^2 + 4b_{02}})y/2$. Moreover it has the first integral

$$H(x, y) = xy - b_{01}xy^2 - \frac{a_{-12}y^3}{3} - b_{02}xy^3 - \frac{a_{-13}y^4}{4}.$$

The linearizing change of the second equation is $z_2 = \ell_1\ell_2^{n_2}\ell_3^{n_3}$, where

$$n_2 = \frac{1}{2}\left(-1 - \frac{b_{01}}{\sqrt{b_{01}^2 + 4b_{02}}}\right) \qquad n_3 = \frac{1}{2}\left(-1 + \frac{b_{01}}{\sqrt{b_{01}^2 + 4b_{02}}}\right),$$

The first equation is linearizable by the transformation $z_1 = H(x,y)/z_2$. As it is mentioned in Section 2 the transformation is analytic. The existence of an analytic linearizing transformation of the first equation follows also from Theorem 7. The conclusion also applies in similar situations below.

In case (2) the system is conjugate to system from case (1). Finally the case (3) that corresponds to case (5) in [22] has a polynomial linearizing change of degree two. $\square$

5. Linearizability of systems with nonhomogeneous nonlinearities of degrees three and five

In this section we give some sufficient linearizability conditions for a system with nonhomogeneous looking for the linearizing change up to certain degree as is described in Section 2.

Theorem 13. *The Hamiltonian system with homogeneous non-linearities of degrees three and five,*

$$\begin{aligned}\dot{x} =& x - a_{20}x^3 - a_{11}x^2y - 3b_{02}xy^2 - a_{-13}y^3 - a_{40}x^5 - a_{31}x^4y - a_{22}x^3y^2 + \\ &- 2b_{13}x^2y^3 - 5b_{04}xy^4 - a_{-15}y^5, \\ \dot{y} =& - y + b_{3,-1}x^3 + 3a_{20}x^2y + b_{11}xy^2 + b_{02}y^3 + b_{5,-1}x^5 + 5a_{40}x^4y + 2a_{31}x^3y^2 + \\ &+ a_{22}x^2y^3 + b_{13}xy^4 + b_{04}y^5.\end{aligned} \tag{5.1}$$

is linearizable at the origin if one of the following conditions holds:

(1) $a_{11} = a_{22} = a_{31} = b_{02} = b_{04} = b_{13} = a_{-13} = a_{-15} = 0$,
(2) $a_{11} = a_{20} = a_{22} = a_{31} = a_{40} = b_{13} = b_{3,-1} = b_{5,-1} = 0$,
(3) $a_{11} = a_{20} = a_{22} = a_{40} = b_{02} = b_{04} = b_{13} = a_{-13} = a_{-15} = b_{3,-1} = b_{5,-1} = 0$,
(4) $a_{11} = a_{22} = a_{-13} = a_{-15} = b_{02} = b_{04} = b_{13} = a_{31}b_{3,-1} - 2a_{20}^3 =$
$= 2a_{40} - a_{20}^2 = 8b_{5,-1} - 3a_{20}b_{3,-1} = 0$,
(5) $a_{11} = 25a_{20}\sqrt{b_{04}^3} - \sqrt{\frac{2}{3}}b_{13}^2 = 5a_{22}b_{04} - 2b_{13}^2 = 625b_{04}^3 - b_{13}^4$
$= 125b_{3,-1}\sqrt{b_{04}^3} - 2\sqrt{\frac{2}{3}}b_{13}^3 = 3125b_{04}^4 - b_{13}^5 = 0$.

Proof. First we note that component (1) and its conjugate (2) are obtained without any calculations. Namely, case (1) corresponds to the system

$$\begin{aligned}\dot{x} =& x + a_{20}x^2 + a_{40}x^5, \\ \dot{y} =& - y + b_{3,-1}x^3 + 3a_{20}x^2y + b_{5,-1}x^5 + 5a_{40}x^4y.\end{aligned}$$

Clearly, the first equation of the system is linearizable. The change is $z_1 = x(x-b_1)^a(x-b_2)^b(x-b_3)^c(x-b_4)^d/(b_1b_2b_3b_4)$ where a, b, c and d are some numbers and $b_1, \ldots, b_4$ are roots of the polynomial $x + a_{20}x^2 + a_{40}x^5$. In fact, it exists if all the roots $b_1, \ldots, b_4$ are different, that is when the discriminant D of the polynomial is not equal to zero. But since the set of linearizable systems is an algebraic set a linearization should exist also for $D = 0$. Moreover, since the system is Hamiltonian, similarly as above, the second equation is linearizable as well.

We have computed 14 pairs of linearizability quantities but we were not able to compute the irreducible composition for the nonfixed values of parameters, so we used the second method. We looked for a linearization of the first equation in the form (2.17) with φ of degree 8. We obtained components (3)–(5) of the statement of the theorem and a subcomponent of (1). The system in the case (3) has, in fact, the nonlinearities of degree 5 only and it is a subsystem of the system in the case (2) from the Theorem 2 in [8]. Its linearizability is proven in [36,37]. The linearizability for other cases is proven below.

In case (1) the system is

$$\begin{aligned}\dot{x} =& x, \\ \dot{y} =& - y + b_{3,-1}x^3 + b_{5,-1}x^5\end{aligned}$$

with Hamiltonian function

$$H(x,y) = -xy + \frac{1}{4}b_{3,-1}x^4 + \frac{1}{6}b_{5,-1}x^6$$

and linearization for the second equation

$$Y = -\frac{H(x,y)}{x} = y - \frac{1}{4}b_{3,-1}x^3 - \frac{1}{6}b_{5,-1}x^5.$$

In case (2) the system is conjugate to system from case (1).

In case (4) the system is

$$\begin{aligned}\dot{x} =& x - a_{20}x^3 - \frac{1}{2}a_{20}^2x^5 - 2\frac{a_{20}^3}{b_{3,-1}}x^4y,\\ \dot{y} =& -y + b_{3,-1}x^3 + 3a_{20}x^2y + \frac{3}{8}a_{20}b_{3,-1}x^5 + \frac{5}{2}a_{20}^2x^4y + 4\frac{a_{20}^3}{b_{3,-1}}x^3y^2,\end{aligned}$$

with Hamiltonian function

$$H(x,y) = -xy + \frac{1}{4}b_{3,-1}x^4 + a_{20}x^3y + \frac{1}{2}a_{20}^2x^5y + \frac{a_{20}^3}{b_{3,-1}}x^4y^2 + \frac{1}{16}a_{20}b_{3,-1}x^6$$

and linearization

$$\begin{aligned}X =& z(x,y) \ = \ x + \frac{1}{2}a_{20}x^3 + 2\frac{a_{20}^2}{b_{3,-1}}x^2y,\\ Y =& -\frac{H(x,y)}{z(x,y)}.\end{aligned}$$

In case (5) we get two solutions with corresponding systems

$$\begin{aligned}\dot{x} =& x \pm \frac{\sqrt{2}}{25\sqrt{3}}\frac{b_{13}^2}{\sqrt{b_{04}^3}}x^3 \mp \sqrt{6b_{04}}xy^2 \mp 10\sqrt{\frac{2}{3}}\frac{\sqrt{b_{04}^3}}{b_{13}}y^3 - \frac{1}{625}\frac{b_{13}^4}{b_{04}^3}x^5\\ &- 25\frac{b_{13}^3}{b_{04}^2}x^4y - \frac{2}{5}\frac{b_{13}^2}{b_{04}}x^3y^2 - 2b_{13}x^2y^3 - 5b_{04}xy^4 - 5\frac{b_{04}^2}{b_{13}}y^5,\\ \dot{y} =& -y \mp \frac{2\sqrt{2}}{125\sqrt{3}}\frac{b_{13}^3}{\sqrt{b_{04}^5}}x^3 \mp \frac{\sqrt{6}}{25}\frac{b_{13}^2}{\sqrt{b_{04}^3}}x^2y \pm \sqrt{\frac{2}{3}b_{04}}y^3 + \frac{1}{3125}\frac{b_{13}^5}{b_{04}^4}x^5\\ &+ \frac{1}{125}\frac{b_{13}^4}{b_{04}^3}x^4y + \frac{2}{25}\frac{b_{13}^3}{b_{04}^2}x^3y^2 + \frac{2}{5}\frac{b_{13}^2}{b_{04}}x^2y^3 + b_{13}xy^4 + b_{04}y^5,\end{aligned}$$

Hamiltonian functions

$$\begin{aligned}H(x,y) =& -xy \mp \frac{1}{125\sqrt{6}}\frac{b_{13}^3}{\sqrt{b_{04}^5}}x^4 \mp \frac{\sqrt{2}}{25\sqrt{3}}\frac{b_{13}^2}{\sqrt{b_{04}^3}}x^3y + \frac{5}{\sqrt{6}}\frac{\sqrt{b_{04}^3}}{b_{13}}y^4 + \frac{1}{18750}\frac{b_{13}^5}{b_{04}^4}x^6+\\ &+ \frac{1}{625}\frac{b_{13}^4}{b_{04}^3}x^5y + \frac{1}{50}\frac{b_{13}^3}{b_{04}^2}x^4y^2 \pm \frac{\sqrt{2}}{\sqrt{3}}\sqrt{b_{04}}x^3y^3 + \frac{1}{2}b_{13}x^2y^4 + b_{04}xy^5 + \frac{5}{6}\frac{b_{04}^2}{b_{13}}y^6,\end{aligned}$$

and linearizations

$$\begin{aligned}X =& z(x,y) = x \mp \frac{1}{25\sqrt{6}}\frac{b_{13}^2}{\sqrt{b_{04}^3}}x^3 \mp \frac{\sqrt{2}}{5\sqrt{3}}\frac{b_{13}}{\sqrt{b_{04}}}x^2y \mp \frac{\sqrt{2}}{\sqrt{3}}\sqrt{b_{04}}xy^2 \mp \frac{5}{\sqrt{6}}\frac{\sqrt{b_{04}^3}}{b_{13}}y^3,\\ Y =& -\frac{H(x,y)}{z(x,y)}.\end{aligned}$$

□

In the previous theorem we studied systems with only odd degree non-linearities. In the next statement we give a result with quadratic non-linear terms included.

Theorem 14. *A Hamiltonian system with non-linearities of degrees two and five, that is, system*

$$\begin{aligned}\dot{x} &= x - a_{10}x^2 - a_{01}xy - a_{-12}y^2 - a_{40}x^5 - a_{31}x^4y - a_{22}x^3y^2 - a_{13}x^2y^3 - a_{04}xy^4 - a_{-15}y^5,\\ \dot{y} &= -y + b_{2,-1}x^2 + b_{10}xy + b_{01}y^2 + b_{5,-1}x^5 + b_{40}x^4y + b_{31}x^3y^2 + b_{22}x^2y^3 + b_{13}xy^4 + b_{04}y^5,\end{aligned} \tag{5.2}$$

with condition (3.1) is linearizable at the origin if one of the following conditions holds:

(1) $b_{13} = b_{04} = b_{01} = a_{-15} = a_{-12} = a_{22} = 0,$
(2) $a_{31} = a_{40} = a_{10} = b_{5,-1} = b_{2,-1} = a_{22} = 0.$

Proof. The computation of the first 14 pairs of linearizability quantities was successful, in contrast with the computation of the irreducible decomposition of their ideal. For this reason we used the second method. We have searched for the conditions of the linearizability using polynomial change (2.18) of degree six and then used the expansion with `Series` of Mathematica up to degree 6. The obtained system of equations forms an ideal, from which we eliminated the coefficients ϕ_{ij} of (2.18). The irreducible decomposition of the acquired ideal yields the first components from the theorem. The second component is conjugate to the first one.

The system of the first case is written as

$$\begin{aligned}\dot{x} =& x - a_{10}x^2 - a_{40}x^5 - a_{31}x^4y,\\ \dot{y} =& -y + b_{2,-1}x^2 + 2a_{10}xy + b_{5,-1}x^5 + 5a_{40}x^4y + 2a_{31}x^3y^2.\end{aligned} \tag{5.3}$$

We apply the substitution

$$u = x^2y, \quad v = x \tag{5.4}$$

and obtain system

$$\dot{u} = u + b_{2,-1}v^4 + 3a_{40}uv^4 + b_{5,-1}v^7, \qquad \dot{v} = v - a_{10}v^2 - a_{31}uv^2 - a_{40}v^5, \tag{5.5}$$

which has a node at the origin. Its normal form is without resonant terms therefore it is linearizable and after the substitution of a form (3.8) the system is

$$\dot{\xi} = \xi, \quad \dot{\eta} = \eta. \tag{5.6}$$

Then, the substitution

$$X = \eta, \quad Y = \xi\eta^{-2}, \tag{5.7}$$

transforms (5.6) into $\dot{X} = X, \dot{Y} = -Y$. □

6. Conclusion

We have revisited the isochronicity and linearizability of two-dimensional polynomial Hamiltonian systems. We have seen the equivalence between the method of finding irreducible decomposition of the linearizability quantities ideal and the method of looking for a linearizing change in order to find the necessary linearizability conditions. Moreover a computational procedure to obtain the necessary and sufficient linearizability of a polynomial system has been given. We have analyzed some complex planar polynomial Hamiltonian systems with homogeneous and nonhomogeneous nonlinearities and proved the existence of linearizable systems among them. However, in some cases, those systems are not all the families where the origin is linearizable. For some of them we have found necessary conditions for the linearizability with the irreducible decomposition of the linearizability quantities ideal. For instance system (2.13) with homogeneous nonlinearities of degrees four, six and seven. For others, the calculation of the irreducible decomposition was impossible and with the second approach of looking for polynomial linearizing change some results are given. Moreover the equivalence between both methods is also established from certain order of the polynomial change. The topic of suitable methods is therefore still open for further investigations.

Acknowledgments

We thank the referees for the careful reading and valuable suggestions which helped to improve the manuscript. The first and third authors are supported by the Slovenian Research Agency (core research program P1-0306). The second author is partially supported by a MINECO/FEDER grant number PID2020-113758GB-I00 and an AGAUR, Spain (Generalitat de Catalunya) grant number 2017SGR-1276.

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